POTENTIAL THEORY AND A CHARACTERIZATION OF POLYNOMIALS IN COMPLEX DYNAMICS

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ABSTRACT. We obtain a measure theoretical characterization of polynomials among rational functions on $\mathbb{P}^1$, which generalizes a theorem of Lopes. Our proof applies both classical and dynamically weighted potential theory.

1. INTRODUCTION

We are interested in a measure theoretical characterization of polynomials among rational functions f of degree $d > 1$ on $\mathbb{P}^1$. Recall that the Fatou set $F(f)$ of f is the region of normality of iterates $\{f^k; k \in \mathbb{N}\}$ in $\mathbb{P}^1$, which by the definition is open. The Julia set $J(f)$ is the complement of $F(f)$ and it is known to be non-empty. Both $F(f)$ and $J(f)$ are f-invariant. The characterization that we have in mind is provided by the following theorem:

Theorem 1. *Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. Suppose that the point ∞ belongs to a Fatou component D_∞ of f, and that $f(D_\infty) = D_\infty$. Then the following are equivalent*:

(i) *f is a polynomial.*
(ii) *The balanced measure μ_f of f coincides with the harmonic measure ν of D_∞ with pole ∞.*

The probability measure μ_f in Theorem 1 is also known to be the (unique) maximal entropy measure of f, which was constructed by Lyubich [8] and by Freire, Lopes and Mañé [5] (see the next paragraph for more historical remarks). Under the additional assumption that $f(\infty) = \infty$, Theorem 1 was proved by Lopes [9]. The Lopes's theorem was stated earlier in Oba and Pitcher [12, Theorem 6], but proved only partially. Lalley gave a probabilistic proof of Lopes's theorem ([7, §6]). In all those proofs, a key role is played by the same equality, which is a consequence of a pullback formula for the logarithmic potential of balanced measure. We will give a simple and conceptually new proof of both this formula and the key equality in an improved form (Lemma 3.3 and Claim 1), which will enable us to prove Theorem 1. Mañé and da Rocha [10] also studied Lopes's theorem in relation to calculations of the entropy of invariant measures on the Julia set.

Brolin's theorem [3, Theorem 16.1] says that the pullbacks $(f^k)^* \delta_a/d^k$ converge weakly to the harmonic measure $\nu = \nu_\infty$ of D_∞ with pole at ∞ when f is a polynomial and δ_a is the Dirac measure at a non-exceptional point $a \in \mathbb{C}$ of f. As a generalization of Brolin's theorem, the balanced measure μ_f was first obtained as the weak limit of pullbacks $(f^k)^* \delta_a/d^k$ for a rational function f and a non-exceptional point $a \in \mathbb{P}^1$ (by Lyubich [8] and by Freire, Lopes and Mañé [5] independently). However, in this article, we will not use these equidistribution results.

In the next section we recall the definition of measures μ_f and ν. Our proof of Theorem 1 is an application of both classical and dynamically weighted potential theory. For related results on (generalized) polynomial-like maps, see [13], [16].

Notation 1.1. We denote the origin of $\mathbb{C}^2$ by 0. Let $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ be the canonical projection so that $\pi(z_0, z_1) = z_1/z_0$ if $z_0 \neq 0$ and $\pi(z_0, z_1) = \infty$ if $z_0 = 0$. Let $\| \cdot \|$ be the Euclidean norm on $\mathbb{C}^2$, and put $(z_0, z_1) \wedge (w_0, w_1) := z_0 w_1 - z_1 w_0$ on $\mathbb{C}^2 \times \mathbb{C}^2$. A function on $\mathbb{P}^1$ is said to be δ-subharmonic (DSH) if it is locally the difference of two subharmonic functions. We normalize d^c so that $dd^c = (i/\pi) \partial \bar{\partial}$. An important example of a value of the dd^c-operator is the generalized Laplacian of the δ-subharmonic function $\log | \cdot - w|$ ($w \in \mathbb{C}$) on $\mathbb{P}^1$, which equals $dd^c \log | \cdot - w| = \delta_w - \delta_\infty$, where δ_w denotes the Dirac measure at $w \in \mathbb{P}^1$.

2. Rational functions and probability measures on $\mathbb{P}^1$

Balanced measure μ_f. For more details, see [6, §4], [15, §1] and [2, Chapitre VIII].

Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. A lift
$$F(z_0, z_1) = (F_0(z_0, z_1), F_1(z_0, z_1))$$
of f is a non-degenerate homogeneous polynomial endomorphism of algebraic degree d on $\mathbb{C}^2$ in that $\pi \circ F = f \circ \pi$ and $F^{-1}(0) = \{0\}$, and is uniquely determined up to multiplication by a constant in $\mathbb{C}^*$.

The dynamical Green function of F is
$$G^F := \lim_{k \to \infty} \frac{1}{d^k} \log \|F^k\| : \mathbb{C}^2 \to \mathbb{R} \cup \{-\infty\}.$$

This convergence is uniform on $\mathbb{C}^2 \setminus \{0\}$, so G^F is continuous there and plurisubharmonic on $\mathbb{C}^2$. It follows from the definition of G^F that

(2.1) $$d \cdot G^F = G^F \circ F,$$

and from homogeneity of F that for every $p \in \mathbb{C}^2$ and every $c \in \mathbb{C}^*$,

(2.2) $$G^F(c \cdot p) = G^F(p) + \log|c|,$$

(2.3) $$G^{cF}(p) = G^F(p) + \frac{1}{d-1} \log|c|.$$

The function $G^F(1, \cdot)$ is continuous on $\mathbb{C}$ and δ-subharmonic on $\mathbb{P}^1$, and the balanced measure μ_f is defined by the unique probability measure on $\mathbb{P}^1$ satisfying

(2.4) $$dd^c G^F(1, \cdot) = \mu_f - \delta_\infty.$$

It follows from (2.3) that the left-hand side of (2.4) is independent of the choice of F, hence the measure μ_f is well-defined. It is also known that $\operatorname{supp} \mu_f = J(f)$ and $J(f)$ is perfect.

From (2.1), the μ_f is balanced and invariant under f: namely,

$$(2.5) \qquad \frac{f^*\mu_f}{d} = \mu_f = f_*\mu_f.$$

We recall that the pullback $f^*\phi$ of continuous function ϕ on $\mathbb{P}^1$ is defined by $\phi \circ f$, and the push-forward $f_*\phi$ is

$$f_*\phi(z) := \frac{1}{d^k} \sum_{w \in f^{-1}(z)} \phi(w),$$

where the sum takes into account of the multiplicity of f at each w. Both $f^*\phi$ and $f_*\phi$ are continuous on $\mathbb{P}^1$, and we may define respectively the push-forward $f_*\mu$ and the pullback $f^*\mu$ of (Radon) measure μ by duality.

Harmonic measure $\nu = \nu_\infty$. For a finite Borel measure ν on $\mathbb{C}$ with compact support, its logarithmic potential on $\mathbb{C}$ is

$$(2.6) \qquad p_\nu(z) := \int_{\mathbb{C}} \log|z - w| \mathrm{d}\nu(w) = \nu(\mathbb{C}) \log|z| + O(|z|^{-1})$$

as $z \to \infty$, and the logarithmic energy of ν is $I_\nu := \int_{\mathbb{C}} p_\nu \mathrm{d}\nu$. A compact set K in $\mathbb{C}$ is said to be polar if

$$\sup\{I_\nu; \operatorname{supp}\nu \subset K, \nu(\mathbb{C}) = 1\} = -\infty.$$

If K is non-polar, then by Frostman's theorem, there is the unique probability measure $\nu = \nu_K$ (the equilibrium measure of K) which attains the supremum in the above. The measure ν has the support on the exterior boundary $\partial_e K$ of K, and satisfies that $p_\nu \equiv I_\nu$ on $K \setminus E$, where E is a (possibly empty) F_σ polar subset of $\partial_e K$. Moreover, $p_\nu > I_\nu$ on D_∞ by the minimum principle.

For a domain D in $\mathbb{P}^1$ which contains ∞ and whose complement $\mathbb{C} \setminus D$ is non-polar, the harmonic measure $\nu = \nu_\infty$ of D with pole ∞ is determined by $\nu_{\mathbb{C} \setminus D}$ (cf. [14, Theorem 4.3.14]). Under the situation in Theorem 1, we will compute I_{μ_f} and see directly that $I_{\mu_f} > -\infty$, so $\mathbb{C} \setminus D_\infty$ is non-polar (see Lemma 3.2 below). Hence D_∞ admits the harmonic measure $\nu = \nu_\infty (= \nu_{\mathbb{C} \setminus D_\infty})$ with pole at ∞.

Dynamically weighted potential theory. A function

$$\mathbb{C}^2 \times \mathbb{C}^2 \ni (p, q) \mapsto \log|p \wedge q| - G^F(p) - G^F(q) \in \mathbb{R} \cup \{-\infty\}$$

descends to a weighted kernel $\Phi_F(z, w)$ $(p \in \pi^{-1}(z), q \in \pi^{-1}(w))$ on $\mathbb{P}^1$. For a Radon measure μ, its F-potential is a δ-subharmonic function

$$U_{F,\mu}(z) := \int_{\mathbb{P}^1} \Phi_F(z, w) \mathrm{d}\mu(w).$$

It can be computed directly that $\mathrm{dd}^c U_{F,\mu} = \mu - \mu(\mathbb{P}^1)\mu_f$, so the potential U_{F,μ_f} of $\mu = \mu_f$ is harmonic on $\mathbb{P}^1$, and hence constant, say, $U_{F,\mu_f} \equiv V_F$. The constant V_F has been computed as

$$(2.7) \qquad V_F = -\frac{1}{d(d-1)} \log|\operatorname{Res} F|$$

in [4, Theorem 1.5]. We will compute it in a different way in Appendix.

3. A proof of Theorem 1

Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and $F = (F_0, F_1)$ a lift of f. We note that

$$f(z) = F_1(1, z)/F_0(1, z).$$

Put $d_0 := \deg F_0(1, z)$ and $d_1 := \deg F_1(1, z)$, and let a_F, b_F be the coefficients of the maximal degree term of $F_0(1, z), F_1(1, z)$, respectively.

Suppose that a Fatou component D_∞ of f contains ∞, and that $f(D_\infty) = D_\infty$.

Lemma 3.1. *For every $z \in \mathbb{C}$,*

$$p_{\mu_f}(z) = G^F(1, z) - G^F(0, 1).$$

In particular, p_{μ_f} is continuous on $\mathbb{C}$.

Proof. Recall that $U_{F,\mu_f} \equiv V_F$ on $\mathbb{P}^1$. Hence for every $z \in \mathbb{C}$,

$$
\begin{aligned}
p_{\mu_f}(z) &= \int_{\mathbb{C}} \log |z - w| \mathrm{d}\mu_f(w) \\
&= U_{F,\mu_f}(z) + G^F(1, z) + \int_{\mathbb{C}} G^F(1, w)\mathrm{d}\mu_f(w) \\
&= G^F(1, z) + C_F,
\end{aligned}
$$

where we put $C_F := V_F + \int_{\mathbb{C}} G^F(1, w)\mathrm{d}\mu_f(w)$. Hence from (2.2) and (2.6),

$$0 = \lim_{z \to \infty} (p_{\mu_f}(z) - \log |z|) = \lim_{z \to \infty} G^F(1/z, 1) + C_F,$$

so that $C_F = -G^F(0, 1)$. □

The following computation of I_{μ_f} may be of independent interest, and was proved in [12, Theorem 4] under the restrictions $f(\infty) = \infty$ and $d_0 < d - 1$, with no reference to G^F.

Lemma 3.2. *The complement $\mathbb{C} \setminus D_\infty$ of D_∞ is non-polar, and D_∞ admits the harmonic measure $\nu = \nu_\infty$ with pole ∞. The energy I_{μ_f} of μ_f is computed as*

$$e^{I_{\mu_f}} = e^{-2G^F(0,1)} |\operatorname{Res} F|^{\frac{1}{d(d-1)}}.$$

Here $\operatorname{Res} F := a_F^{d-d_1} b_F^{d-d_0} R(F_0(1, z), F_1(1, z))$ is the homogeneous resultant of F, where $R(P(z), Q(z))$ is the resultant of two polynomials $P(z)$ and $Q(z)$.

Proof. Integrating the equality in Lemma 3.1 in $\mathrm{d}\mu_f(z)$, we get

$$I_{\mu_f} = \int_{\mathbb{C}} p_{\mu_f} \mathrm{d}\mu_f = \int_{\mathbb{C}} G^F(1, \cdot)\mathrm{d}\mu_f - G^F(0, 1)(> -\infty),$$

which with $\operatorname{supp} \mu_f \subset J(f) \subset \mathbb{C} \setminus D_\infty$ implies that $\mathbb{C} \setminus D_\infty$ is non-polar. It also follows from the proof of the previous lemma that

$$-G^F(0, 1) = C_F = V_F + \int_{\mathbb{C}} G^F(1, \cdot)\mathrm{d}\mu_f,$$

where C_F has been introduced in the proof of the previous lemma. The value of V_F has already been computed in (2.7). Hence

$$I_{\mu_f} = \frac{1}{d(d-1)} \log |\operatorname{Res} F| - 2G^F(0, 1).$$

□

The following was proved in [12, p307] and [9, p398] under the assumption $f(\infty) = \infty$ and $z \in J(f)$ (again with no reference to G^F).

Lemma 3.3. *For every* $z \in \mathbb{C} \setminus f^{-1}(\infty)$,

$$p_{\mu_f}(f(z)) = d \cdot p_{\mu_f}(z) - \log|F_0(1, z)| + (d-1)G^F(0, 1).$$

Proof. For every $z \in \mathbb{C} \setminus f^{-1}(\infty)$, from (2.1),

$$G^F(1, f(z)) = G^F(F(1, z)) - \log|F_0(1, z)| = d \cdot G^F(1, z) - \log|F_0(1, z)|.$$

Now Lemma 3.1 completes the proof. □

We now proceed with the proof of our theorem. Let us denote the harmonic measure of D_∞ with pole at ∞ by $\nu = \nu_\infty$.

(ii)$\Rightarrow$(i). Suppose first that $\mu_f = \nu$. Under this assumption, $p_\nu = p_{\mu_f}$ is continuous on $\mathbb{C}$ (Lemma 3.1), so we have $p_{\mu_f} = p_\nu \equiv I_\nu = I_{\mu_f}$ on $\mathbb{C} \setminus D_\infty$.

A weaker form of the identity in Claim 1 was the key equality which we mentioned in §1 ([9, p398], [7, Claim 2 in §6]).

Claim 1. *On* $\mathbb{C} \setminus f^{-1}(D_\infty)$,

$$|F_0(1, \cdot)| \equiv e^{(d-1)(I_{\mu_f} + G^F(0,1))}.$$

Proof. The assumption $f(D_\infty) = D_\infty$ implies $D_\infty \subset f^{-1}(D_\infty)$. Hence $p_{\mu_f} \circ f = p_{\mu_f} \equiv I_{\mu_f}$ on $\mathbb{C} \setminus f^{-1}(D_\infty)$. Now Lemma 3.3 completes the proof. □

Suppose that $F_0(1, \cdot)$ is non-constant, that is, $d_0 > 0$. Our refinement of the key equality make the following reduction possible.

Claim 2. $F(f) = D_\infty$.

Proof. If $F(f) \neq f^{-1}(D_\infty)$, then $F(f) \setminus f^{-1}(D_\infty)$ is a non-empty open subset of $\mathbb{C} \setminus f^{-1}(D_\infty)$. By the identity theorem, Claim 1 implies that $F_0(1, \cdot)$ must be constant, which contradicts the assumption $d_0 > 0$. Hence $F(f) = f^{-1}(D_\infty)$, which implies that $F(f) = f(F(f)) = D_\infty$. □

By Claim 1, $J(f)$ is contained in the lemniscate

$$L := \{z \in \mathbb{C}; |F_0(1, z)| = e^{(d-1)(I_{\mu_f} + G^F(0,1))}\}.$$

We note that each component of L is a (possibly non-simple) closed curve in $\mathbb{C} \setminus f^{-1}(\infty)$, which is real-analytic except for finitely many singularities; more precisely, for each $z_0 \in L$, there are an $n \in \mathbb{N}$, a Möbius transformation K and a local holomorphic coordinate h around z_0 such that $h(z_0) = 0$, $K(F_0(1, z_0)) = 0$, $\mathrm{Im}K(F_0(1, \cdot)) \equiv 0$ on L and $K(F_0(1, h^{-1}(w))) = w^n$ around 0. In particular, L around z_0 is the image of $\bigcup_{-n \leq j < n}\{w; \arg w = j\pi/n\} \cup \{0\}$ under h^{-1}, and $n \geq 2$ if and only if z_0 is a critical point of $F_0(1, \cdot)$.

Fix a component l of L intersecting $J(f)$.

Claim 3. *For every* $k \in \mathbb{N}$, $f^k(l) \subset L$.

Proof. Fix $z_0 \in l \cap J(f)$. Since L has at most d_0 components, there exists $\delta > 0$ such that $\{|z - z_0| < \delta\} \cap J(f) \subset l$. Hence from the perfectness of $J(f)$, z_0 is a non-isolated point of $l \cap J(f)$.

For every $k \in \mathbb{N}$, let L_k be the component of L containing $f^k(z_0)$. By the same argument as the above, there is $\delta_k > 0$ such that $\{|z - f^k(z_0)| < \delta_k\} \cap J(f) \subset L_k$.

Hence if $\delta > 0$ is small enough, then $f^k(\{|z - z_0| < \delta\} \cap l \cap J(f)) \subset L_k$. Then by an argument involving the identity theorem [11], $f^k(l) \subset L_k(\subset L)$. $\qquad \square$

Claim 4. $l \subset J(f)$.

Proof. Suppose that $l \cap F(f) \neq \emptyset$. Then there exists $z_0 \in l \cap D_\infty$ by Claim 2, and hence $p_{\mu_f}(z_0) = p_\nu(z_0) > I_\nu = I_{\mu_f}$. By Claim 3, for every $k \in \mathbb{N}$, we have $f^{k-1}(z_0) \subset L \subset \mathbb{C} \setminus f^{-1}(\infty)$, which with Lemma 3.3 (and the definition of L) implies that $p_{\mu_f}(f^k(z_0)) - I_{\mu_f} = d \cdot (p_{\mu_f}(f^{k-1}(z_0)) - I_{\mu_f})$. Hence

$$p_{\mu_f}(f^k(z_0)) - I_{\mu_f} = d^k \cdot (p_{\mu_f}(z_0) - I_{\mu_f}) > 0,$$

so $\lim_{k\to\infty} p_{\mu_f}(f^k(z_0)) = \infty$. On the other hand, by Claim 3, we have $(f^k(z_0)) \subset L$, and since p_{μ_f} is upper semicontinuous, we have $\sup_{k\in\mathbb{N}} p_{\mu_f}(f^k(z_0)) \leq \sup_L p_{\mu_f} < \infty$. This is a contradiction. $\qquad \square$

Let U be a component of $\mathbb{P}^1 \setminus l$ not containing ∞. Then $\partial U \subset J(f)$. By the maximum modulus principle, it follows that $U \subset \mathbb{C} \setminus L$, and from $J(f) \subset L$, we have $U \subset F(f)$. Hence U is a Fatou component of f, and by Claim 2, we must have $U = F(f) = D_\infty$. This contradicts our assumption $\infty \notin U$.

Now the proof of (ii)$\Rightarrow$(i) is complete.

(i)$\Rightarrow$(ii). Here we will give a proof of the assertion $\mu_f = \nu$ without the equidistribution results mentioned in Section 1, using only computation of capacity.

Suppose that f is a polynomial, or equivalently, that $F_0(1, z) \equiv a_F$ on $\mathbb{C}$. By a direct computation, Lemma 3.2 implies that

$$e^{I_{\mu_f}} = \exp\left(-2 \cdot \frac{1}{d-1} \log|b_F|\right) \cdot (|a_F b_F|^d)^{\frac{1}{d(d-1)}} = |a_F/b_F|^{\frac{1}{d-1}}.$$

Brolin's theorem which we mentioned in Section 1 was based on the computation of e^{I_ν} ([3, Lemma 15.1]) as

$$e^{I_\nu} = |b_F/a_F|^{-\frac{1}{d-1}}.$$

Hence $I_{\mu_f} = I_\nu$, and from the uniqueness of $\nu = \nu_{\mathbb{C}\setminus D_\infty}$, we have $\mu_f = \nu$.

Now the proof of Theorem 1 is complete. $\qquad \square$

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4. APPENDIX: A COMPUTATION OF V_F

Let f be a rational function of degree $d > 1$, and $F = (F_0, F_1)$ be a lift of f. For completeness, we give a direct computation (2.7) of V_F, again without using the equidistribution theorem. Indeed, we can give a proof of the equidistribution theorem based on (2.7). For the original computation of V_F, which uses the equidistribution theorem, see DeMarco [4, Theorem 1.5]. After having written this appendix, we learned that similar computations and formulas appeared in Appendix A in [1].

We continue to use the notation d_0, d_1, a_F, b_F as in §3. Let us write as $F_0(1, z) = a_F \prod_{j=1}^{d_0} (z - w_j)$. Then since $F : \mathbb{C}^2 \to \mathbb{C}^2$ is homogeneous and of topological degree

d^2, we get on $\mathbb{C}^2$,

$$|F(p) \wedge (0,1)| = |F_0(p)| = |a_F||p \wedge (0,1)|^{d-d_0} \prod_j |p \wedge (1, w_j)|$$

$$= (|a_F|(|F_1(0,1)|^{1/d})^{d-d_0} \prod_j |F_1(1, w_j)|^{1/d}) \prod_{q \in F^{-1}(0,1)} |p \wedge q|^{1/d},$$

and the leading coefficient is computed as

$$|a_F|(|F_1(0,1)|^{1/d})^{d-d_0} \prod_j |F_1(1, w_j)|^{1/d}$$

$$= (|a_F|^d |b_F|^{d-d_0} \cdot |a_F|^{-d_1} |R(F_0(1, z), F_1(1, z))|)^{1/d} = |\operatorname{Res} F|^{1/d}.$$

Hence on $\mathbb{C}^2$,

$$(4.1) \qquad |F(p) \wedge (0,1)| = |\operatorname{Res} F|^{1/d} \prod_{q \in F^{-1}(0,1)} |p \wedge q|^{1/d}.$$

From (2.1), $G^F(F(p)) = d \cdot G^F(p)$ and $G^F(q) = G^F(0,1)/d$ ($q \in F^{-1}(0,1)$). Hence the log of (4.1) descends to $\mathbb{P}^1$ as

$$\Phi_F(f(z), \infty) = \frac{1}{d} \log |\operatorname{Res} F| + \int_{\mathbb{P}^1} \Phi_F(z, w) \mathrm{d}(f^* \delta_\infty)(w).$$

Integrating this in $\mathrm{d}\mu_f(z)$,

$$\int_{\mathbb{P}^1} \Phi_F(f(z), \infty) \mathrm{d}\mu_f(z) = \frac{1}{d} \log |\operatorname{Res} F| + \int_{\mathbb{P}^1} U_{F, \mu_f}(w) \mathrm{d}(f^* \delta_\infty)(w),$$

and from $f_* \mu_f = \mu_f$ and $U_{F, \mu_f} \equiv V_F$,

$$V_F = \int_{\mathbb{P}^1} \Phi_F(\infty, \cdot) \mathrm{d}f_* \mu_f = \int_{\mathbb{P}^1} \Phi_F(\infty, f(z)) \mathrm{d}\mu_f(z)$$

$$= \frac{1}{d} \log |\operatorname{Res} F| + \int_{\mathbb{P}^1} V_F \mathrm{d}(f^* \delta_\infty)(w) = \frac{1}{d} \log |\operatorname{Res} F| + d \cdot V_F.$$

Now the proof of (2.7) is completed. $\qquad \square$

Remark 4.2. From (4.1),

$$1 = \prod_{p \in F^{-1}(1,0)} |F(p) \wedge (0,1)| = |\operatorname{Res} F|^{(1/d) \cdot d^2} \prod_{p \in F^{-1}(1,0), q \in F^{-1}(0,1)} |p \wedge q|^{1/d},$$

and we also obtain an important formula

$$|\operatorname{Res} F| = \prod_{p \in F^{-1}(1,0), q \in F^{-1}(0,1)} |p \wedge q|^{-1/d^2}.$$

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